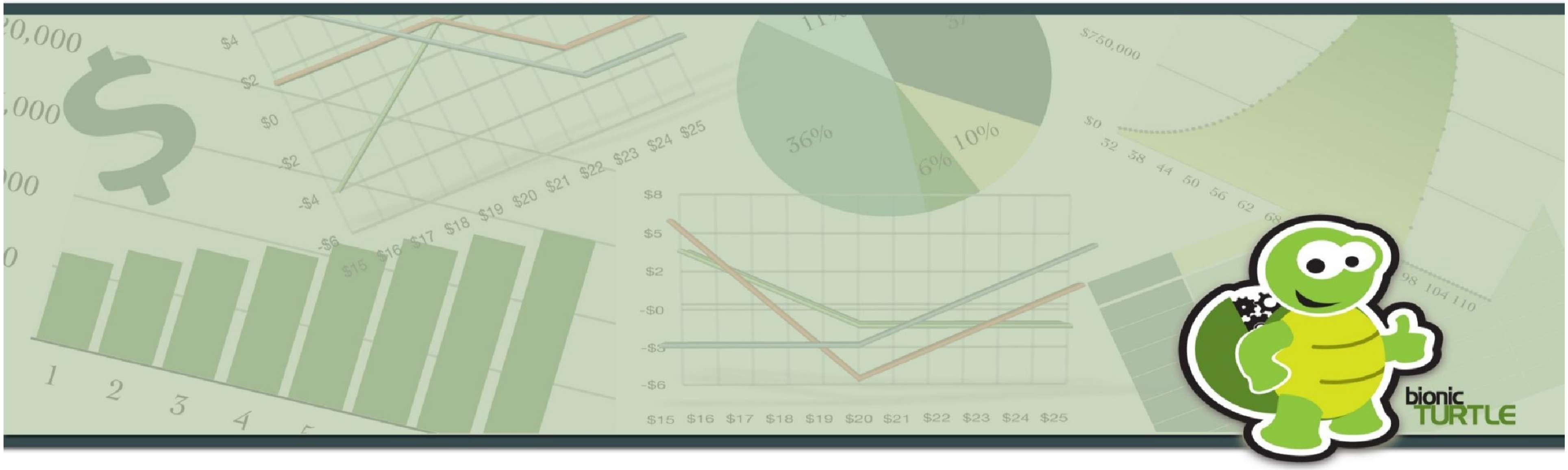


P1.T3. Financial Markets & Products

Hull, Options, Futures & Other Derivatives
Interest Rates

Bionic Turtle FRM Video Tutorials

By David Harper, CFA FRM

Interest Rates

- Describe Treasury Rates, LIBOR, Repo Rates, and what is meant by the risk-free rate.
- Calculate the value of an investment using different compounding frequencies.
- Convert interest rates based on different compounding frequencies
- Calculate the theoretical price of a bond using spot rates.
- Derive forward interest rates from a set of spot rates.
- Derive the value of the cash flows from a forward rate agreement (FRA).
- Calculate the duration, modified duration and dollar duration of a bond.
- Evaluate the limitations of duration and explain how convexity addresses some of them.
- Calculate the change in a bond’s price given duration, convexity, and a change in interest rates.
- Compare and contrast the major theories of the term structure of interest rates.

BIONIC TURTLE

Describe Treasury Rates, LIBOR, Repo Rates, and what is meant by the risk-free rate.

Treasury Rates & LIBOR

Treasury rates are the rates an investor earns on Treasury bills and Treasury bonds; i.e., **government borrowing in its own currency**.

- Treasury rates are totally risk-free rates in the sense that an investor who buys a Treasury bill or Treasury bond is certain that interest and principal payments will be made as promised.

LIBOR (London Interbank Offered Rate)

A LIBOR quote by a bank is the rate of interest at which the bank is prepared to make an unsecured large wholesale deposit with other banks.

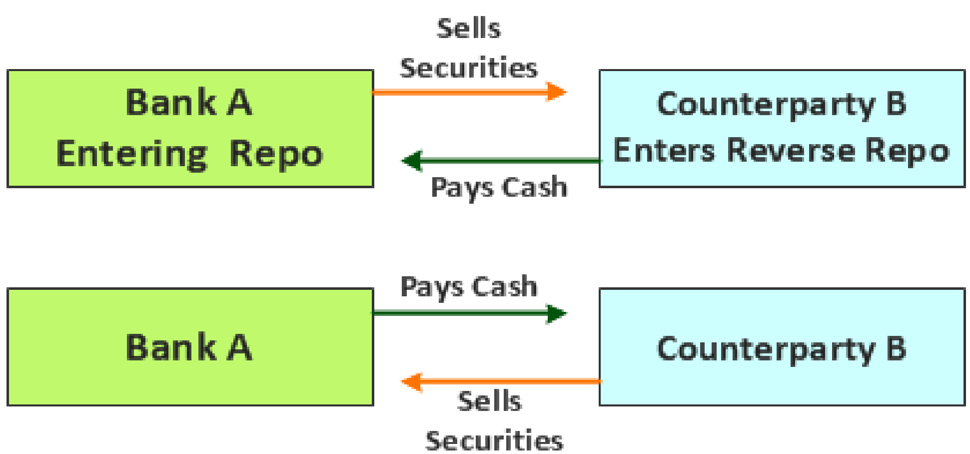
- Large banks and other financial institutions quote LIBOR in all major currencies for maturities up to 12 months: 1-month LIBOR is the rate at which 1-month deposits are offered, 3-month LIBOR is the rate at which 3-month deposits are offered, and so on.

BIONIC TURTLE

Describe Treasury Rates, LIBOR, Repo Rates, and what is meant by the risk-free rate (continued)

Repo Rates

- Repurchase agreement ("repo"): a contract where dealer agrees to sell securities to counterparty now and buy them back later at a higher price.
  - ❑ The counterparty is lending a collateralized (secured) loan.
  - ❑ Very little credit risk. If the borrower defaults, lender keeps the securities
- **Repo rate**: difference between selling price (today) and the repurchased price (later) is the interest paid in obtaining a loan
- ***The most common type of repo is an overnight repo***.



Describe Treasury Rates, LIBOR, Repo Rates, and what is meant by the risk-free rate (continued)

Risk-Free Rate

Risk-Free Rate

- **Prior to the crisis**, LIBOR rates were short-term risk-free rates. For a AA-rated financial institution, LIBOR is the short-term opportunity cost of capital.

OIS is modern riskfree rate

- **Since the crisis**, Overnight Index Swap (OIS) rates have been used as risk-free rates
- **Treasury rates are too low** to be used as risk-free rates because:
  - ***Market demand***: To fulfill regulatory requirements, FIs must purchase Treasury bills/bonds. This **increases demand** for Treasury instruments.
  - ***Regulatory relief***: The amount of (regulatory) capital required to support an investment in Treasury bills/bonds is smaller than the capital required to support a similar investment in other instruments.
  - ***Tax treatment***: In the United States, Treasury instruments are given a favorable tax treatment because they are not taxed at the state level.

Calculate the value of an investment using different compounding frequencies.

If we assume that A is the amount invested, R_c is the rate of interest with continuous compounding, R_m is the equivalent rate of interest with discrete compounding (m times per annum), and n is the number of years, then:

$$\boldsymbol{Ae^{R_c n} = A\left(1+\frac{R_m}{m}\right)^{mn} \qquad \rightarrow e^{R_c} = \left(1+\frac{R_m}{m}\right)^{m}}$$

Therefore:

- We can translate *from a continuous to discrete* rate with:

$$\boldsymbol{R_m = m\left(e^{R_c/m} - 1\right)}$$

- We can translate *from a discrete to a continuous* rate with:

$$\boldsymbol{R_c = m\ln\left(1+\frac{R_m}{m}\right)}$$

BIONIC TURTLE

Calculate the value of an investment using different compounding frequencies (continued)

Here we see the effect of the compounding frequency on the value of $100 at the end of 1 year when the interest rate is 10% per annum.

Initial value	**$100.00**
Stated (aka, nominal) rate	**10.0%**
Number of years	**1.0**

Compound frequency		**Value of Value of $100.00 at end of period**	
m =			
1	Annual	$110.0000	= $100.00*(1+10.0%/1)^(1.0*1)
2	Semiannual	110.2500	= $100.00*(1+10.0%/2)^(1.0*2)
4	Quarterly	110.3813	= $100.00*(1+10.0%/4)^(1.0*4)
12	Monthly	110.4713	= $100.00*(1+10.0%/12)^(1.0*12)
52	Weekly	110.5065	= $100.00*(1+10.0%/52)^(1.0*52)
365	Daily	110.5156	= $100.00*(1+10.0%/365)^(1.0*365)
e = 2.71828	Continuous	110.5171	= $100.00*exp(10.0%*1.0)

Convert interest rates based on different compounding frequencies

The present value is discretely discounted at (m) periods per year (e.g., m = 2 for semi-annual compounding) over (n) years by using the formula on the left. The continuous equivalent is the right. If the future value is one dollar (FV = $1), then the PV is the discount factor (DF).

Discrete	Continuous
$PV = \frac{FV}{\left(1+\frac{r}{m}\right)^{mn}}$	$PV = FVe^{-rn}$
$PV = \frac{\$1}{\left(1+\frac{r}{m}\right)^{mn}}$	$PV = \$1e^{-rn}$

Discount Factor (DF), 10 years @ 8% semi-annual	Discount Factor, 10 years @ 8% continuous
$DF(PV) = \frac{\$1}{\left(1+\frac{8\%}{2}\right)^{2*10}}$	$DF(PV) = \$1e^{-8\%*10}$
$DF = 0.4564$	$DF = 0.4493$

BIONIC TURTLE

Convert interest rates based on different compounding frequencies (continued)

We also must be able to convert from a discrete rate into a continuous rate, and vice-versa:

Semi-annual equivalent of 8% continuous:	Continuous equivalent of 8.162% semi-annual:
$R_m = m(e^{R_c/m} - 1)$	$R_c = m \ln\left(1 + \frac{R_m}{m}\right)$
$R_m = 2(e^{8\%/2} - 1) = 8.162\%$	$R_c = 2 \ln\left(1 + \frac{8.162\%}{2}\right) = 8\%$

BIONIC TURTLE

Convert interest rates based on different compounding frequencies (continued)

Discrete Rate R(m)	Periods per year (k)	Continuous Rate R(c)	
8.00%	1	7.696%	= 1*LN(1+ 0.0800/1)
8.00%	2	7.844%	= 2*LN(1+ 0.0800/2)
8.00%	4	7.921%	= 4*LN(1+ 0.0800/4)
8.00%	12	7.973%	= 12*LN(1+ 0.0800/12)

Continuous Rate R(c)	Periods per year (k)	Discrete Rate R(m)	
8.00%	1	8.329%	= 1*[EXP(0.0800/1) -1]
8.00%	2	8.162%	= 2*[EXP(0.0800/2) -1]
8.00%	4	8.081%	= 4*[EXP(0.0800/4) -1]
8.00%	12	8.027%	= 12*[EXP(0.0800/12) -1]

Calculate the theoretical price of a bond using spot rates.

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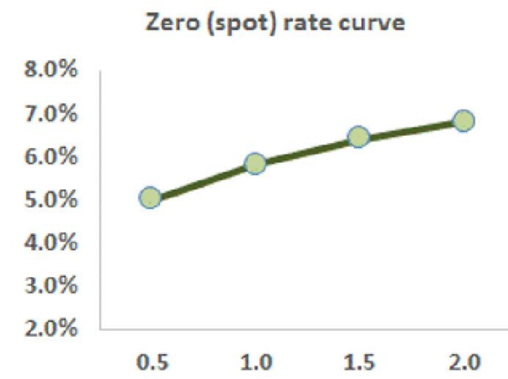
To calculate the price of a coupon-paying bond, each cash flow is discounted by the appropriate discount factor (or, equivalently, by using the corresponding spot rate).

Example: Consider the situation where Treasury zero rates, measured with continuous compounding, are as shown in the table. Suppose that a 2-year Treasury bond with a principal of $100.00 provides semi-annual coupons at the rate of 6.0% per annum.

Hull Table 4.2: Treasury zero rates

Face value (aka, principal, par)	$100.00
Semi-annual coupon (per annum)	6.0%

Maturity (yrs)	Zero Rates (CC)	Cash Flows FV	Cash Flows PV
0.5	5.0%	$3.00	$2.93
1.0	5.8%	$3.00	$2.83
1.5	6.4%	$3.00	$2.73
2.0	6.8%	$103.00	$89.90
		Price (sum)	$98.39



Notes:

1. Coupon rate is always per annum; eg, 6.0% semi-annual (s.a.) coupon pays 3.0% every six months
2. "CC" refers to continously compounded zero rate; e.g., $2.93 = $3.00*exp(-0.050*0.5)

Calculate the theoretical price of a bond using spot rates (continued)

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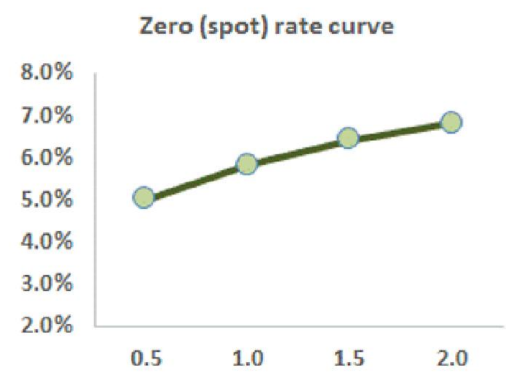
To calculate the price of a coupon-paying bond, each cash flow is discounted by the appropriate discount factor (or, equivalently, by using the corresponding spot rate).

- Coupon rate is always per annum; e.g. 6.0% semi-annual (s.a.) coupon pays 3.0% every six months
- Given the zero rate curve, for eg. the 6 month zero rate is 5.0%. "CC" refers to continuously compounded zero rate. Under continuous compounding, the present value (PV) of the coupon cash flow of $3.00 is $2.93 = $3.00 * EXP(-0.050*0.5).

Hull Table 4.2: Treasury zero rates

Face value (aka, principal, par)	$100.00
Semi-annual coupon (per annum)	6.0%

Maturity (yrs)	Zero Rates (CC)	Cash Flows FV	Cash Flows PV
0.5	5.0%	$3.00	$2.93
1.0	5.8%	$3.00	$2.83
1.5	6.4%	$3.00	$2.73
2.0	6.8%	$103.00	$89.90
		Price (sum)	$98.39



Notes:
1. Coupon rate is always per annum; eg, 6.0% semi-annual (s.a.) coupon pays 3.0% every six months
2. "CC" refers to continously compounded zero rate; e.g., $2.93 = $3.00*exp(-0.050*0.5)

- Likewise, each cash flow is discounted by its respective zero rate. The theoretical model price ($98.39) is the sum of the present values (PVs) of these cash flows.

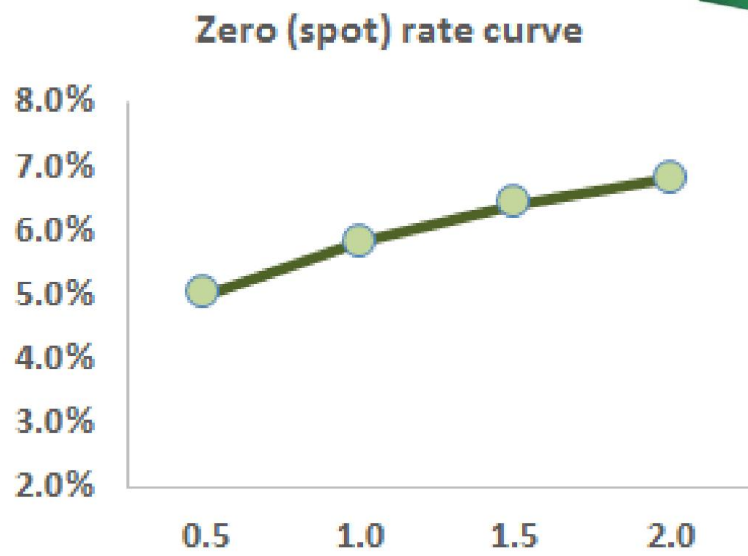
BIONIC TURTLE

Calculate the theoretical price of a bond using spot rates (continued)

Hull Table 4.2

Face value (aka, principal, par)	$100.00
Semi-annual coupon (per annum)	6.0%

Maturity (yrs)	Zero Rates (CC)	Cash Flows FV	Cash Flows PV
0.5	5.0%	$3.00	$2.93
1.0	5.8%	$3.00	$2.83
1.5	6.4%	$3.00	$2.73
2.0	6.8%	$103.00	$89.90
		Price (sum)	$98.39



Notes:

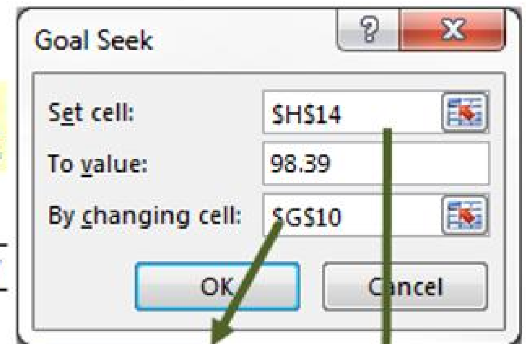
1. Coupon rate is always per annum; eg, 6.0% semi-annual (s.a.) coupon pays 3.0% every six months
2. "CC" refers to continously compounded zero rate; e.g., $2.93 = $3.00*exp(-0.050*0.5)

Calculate the theoretical price of a bond using spot rates (continued)

A bond's yield (YTM) is the single rate that discounts the bond's cash flows to match its market price. Unlike the zero-rate curve (which it averages), it is effectively a flat line.

Suppose $98.39 is the observed market price of the bond. We can use an iterative procedure to retrieve the *single discount rate* that, when applied to all cash flows, returns a bond price equal to this market price. The single rate that discounts (then sums) all cash flows to a price of $98.39 is found to be 6.76%, as shown in the table below.

Hull sec 4.4: Bond Yield Example



Face value; aka, principal, par	$100.00
Semi-annual coupon (per annum)	6.0%

Maturity (yrs)	Zero Rates (CC)	Cash Flows FV	Cash Flows PV		
0.5	5.0%	$3.00	$2.93	6.76%	$2.90
1.0	5.8%	$3.00	$2.83	6.76%	$2.80
1.5	6.4%	$3.00	$2.73	6.76%	$2.71
2.0	6.8%	$103.00	$89.90	6.76%	$89.98
		Price (sum)	$98.39		$98.39

Note:
The yield (YTM) is the single rate that discounts the bond's cash flows to match its market price. Unlike the zero rate curve (which it averages), it is effectively a flat line.

BIONIC TURTLE

Calculate the theoretical price of a bond using spot rates (continued)

Bond Yield

Semi-annual yield with calculation

Face value; aka, principal, par	$100.00	YTM:	4.944%
Semi-annual coupon (per annum)	6.0%	=RATE(B15*2,E5*E6/2,-E16,E5)*2	

Maturity (yrs)	Zero Rates (CC)	Cash Flows FV	Cash Flows PV		
0.5	2.0%	$3.00	$2.97	4.94%	$2.93
1.0	3.6%	$3.00	$2.89	4.94%	$2.86
1.5	4.4%	$3.00	$2.81	4.94%	$2.79
2.0	5.0%	$103.00	$93.31	4.94%	$93.42
		Price (sum)	$101.988		$101.988

BIONIC TURTLE

Calculate the theoretical price of a bond using spot rates (continued)

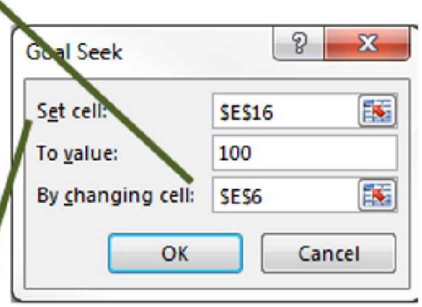
Par Yield

The par yield for a bond is the coupon rate that causes the bond price to equal its principal or par value. **The par yield discounts cash flows at the various zero rates, but finds the coupon rate that discounts the present value (model price) to its par value (i.e, $100.00 or $1,000.00**)

Hull sec 4.4: Par Yield Example

Face value; aka, principal, par	$100.00
Semi-annual coupon (per annum)	6.873%

Maturity (yrs)	Zero Rates (CC)	Cash Flows FV	Cash Flows PV
0.5	5.0%	$3.44	$3.35
1.0	5.8%	$3.44	$3.24
1.5	6.4%	$3.44	$3.12
2.0	6.8%	$103.44	$90.28
		Price (sum)	$100.00



Note:
The par yield discounts cash flows at the various zero rates, but finds the copuon rate that discounts the present value (model price) to par of $100.00 (or $1,000.00)

Using the same zero rates in Table 4.2, the value of the bond is equal to its par value of 100 when by iteration we find the coupon rate or the 2-year par yield is 6.87% per annum in case of semiannual compounding.

Calculate the theoretical price of a bond using spot rates (continued)

Par Yield

With formula …

Hull sec 4.4: par yield with formula

Face value; aka, principal, par	$100.00
Semi-annual coupon (per annum)	6.873%

Maturity (yrs)	Zero Rates (CC)	df(t)
0.5	5.0%	0.9753
1.0	5.8%	0.9436
1.5	6.4%	0.9085
2.0	6.8%	0.8728
	A =	3.70027
	c =	6.8729%

Cash Flows	
FV	PV
$3.44	$3.35
$3.44	$3.24
$3.44	$3.12
$103.44	$90.28
Price (sum)	$100.00

$$c = \frac{(1-d)m}{A} = \frac{(1-0.8728)2}{3.70027}$$

BIONIC TURTLE

Calculate the theoretical price of a bond using spot rates (continued)

Bootstrapping

Determining Treasury Spot rates

Now that we have seen the calculation of bond rates and par yields, let's calculate Treasury zero rates like those in Hull Table 4.2 which are used to obtain the theoretical price of a bond. The most popular approach to determine them is known as the **bootstrap method**.

- **Example:** To illustrate the bootstrap, consider the following data on the prices of five bonds.

Hull Table 4.3 & 4.4: Determining Treasury Zero rates; aka, bootstrap

Face value	$100.00				
Years to Maturity	0.25	0.5	1.0	1.5	2.0
Bond price (PV)	$97.50	$94.90	$90.00	$96.00	$101.60
Coupon rate	0.00%	0.00%	0.00%	8.00%	12.00%
Spot rates	10.127%	10.469%	10.536%	10.681%	10.808%

0.25 Yrs	CF:	$100.00	n.a	n.a	n.a	n.a
	rate:	10.127%				
0.50 Yrs	CF:		$100.00	$0.00	$4.00	$6.00
	rate:		10.469%			
1.00 Yrs	CF:			$100.00	$4.00	$6.00
	rate:			10.536%		
1.50 Yrs	CF:				$104.00	$6.00
	rate:				10.681%	
2.00 Yrs	CF:					$106.00
	rate:					10.808%

BIONIC TURTLE

Calculate the theoretical price of a bond using spot rates (continued)

Bootstrapping

- Since the first three bonds pay no coupons, the zero rates corresponding to the maturities of these bonds can easily be calculated. For example, the 3-month bond has the effect of turning an investment of $97.50 into $100.00 in 3 months such that: $100 = 97.5e^{r*0.25}$. So, the continuously compounded 3-month spot rate r is calculated as:

$$\ln(100/97.5) \times (1/0.25) = 10.127\%.$$

- The other two spot rate can be calculated as follows. The fourth bond has two payments of $4.00 coupon, one in six months and another in one year. In 1.5 years, it gets the final principal payment of $100.00 along with the $4.00 coupon payment. From the earlier calculations (hence the *bootstrap*), we know the discount rate for the payment at the end of six months is 10.469% and the discount rate for the payment at the end of 1 year is 10.536%. We also know that the bond's price, $96.00, must equal the present value of all the payments received by the bondholder such that:

$$96 = 4e^{-0.10469*0.5} + 4e^{-0.10536*1} + 104e^{-r*1.5} \rightarrow e^{-r*1.5} = 0.85196$$

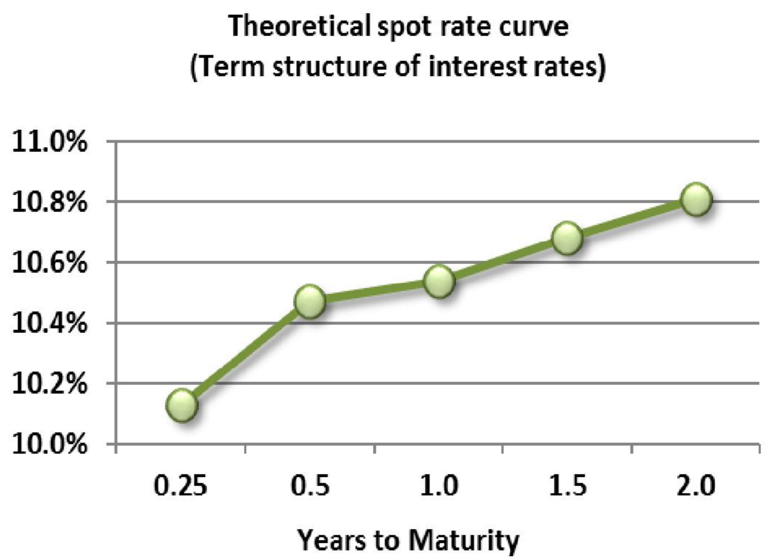
- Therefore, $r = -\ln(0.85196)/1.5$

BIONIC TURTLE

Calculate the theoretical price of a bond using spot rates (continued)

Bootstrapping

- Solving this, the 1.5-year spot is calculated to be 10.681%.
- Likewise, the 2-year spot rate is calculated as 10.808%.
- The treasury zero rates determined by this bootstrap method are plotted in the graph at the right.



BIONIC TURTLE

Derive forward interest rates from a set of spot rates.

Hull assumes a continuous compounding/discounting frequency. The forward (continuously compounded) rate, R_F, is given by:

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

For example, assume this zero rate curve and the implied one-year forward rates:

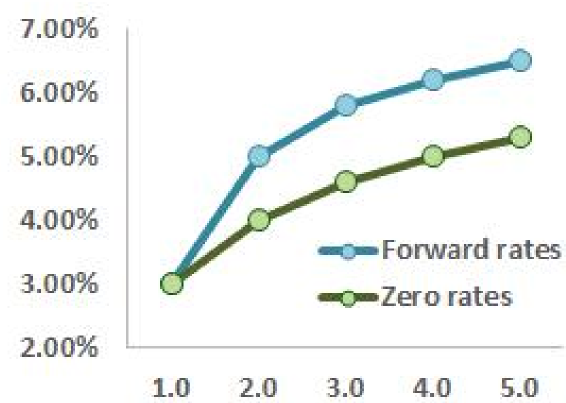
- **The one-year forward rate for year two:**

$$\frac{[4\% \times 2 - 3\% \times 1]}{[2 - 1]} = 5.0\%$$

- **The one-year forward rate for year four:**

$$\frac{[5\% \times 4 - 4.6\% \times 3]}{[4 - 3]} = 6.2\%$$

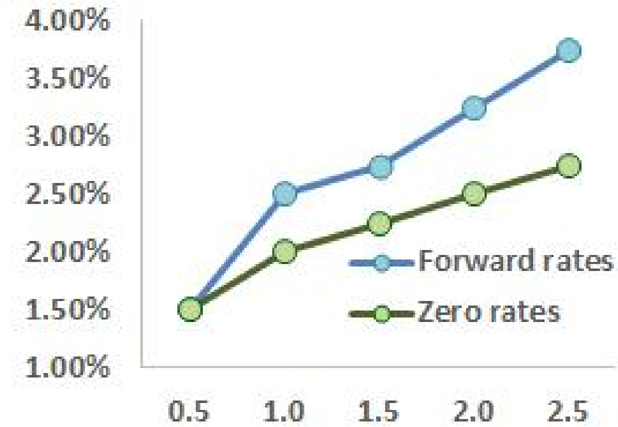
Maturity (yrs)	Zero Rates (CC)	One-year Forward
1.0	3.00%	3.00%
2.0	4.00%	5.00%
3.0	4.60%	5.80%
4.0	5.00%	6.20%
5.0	5.30%	6.50%



Derive forward interest rates from a set of spot rates (continued)

Here, 6-month forward rates are extracted from the spot rate curve.

Maturity (yrs)	Zero Rates (CC)	6 month Forward
0.5	1.50%	1.50%
1.0	2.00%	2.50%
1.5	2.25%	2.75%
2.0	2.50%	3.25%
2.5	2.75%	3.75%



- It is good practice to extract these forward rates. For example, given the same zero rate curve, what is the six-month ***continuous*** forward rate starting in 1.5 years, $F_{1.5,2}$?

$$\frac{[2.5\% \times 2 - 2.25\% \times 1.5]}{[2 - 1.5]} = 3.25\%$$

- Another. What is the six-month ***semi-annual*** forward rate starting in 1.5 years, $F_{1.5,2}$?

$$\left[\frac{(1+2.5\%/2)^{2*2}}{(1+2.25\%/2)^{1.5*2}} - 1\right] = 3.252\%$$

Derive the value of the cash flows from a forward rate agreement (FRA).

A forward rate agreement (FRA) is an agreement that a fixed interest rate will apply to either borrowing or lending a certain principal during a specified future time period.

- An FRA is equivalent to an agreement where interest at a predetermined rate, R_k is exchanged for interest at the market rate (LIBOR).
- An FRA is valued by assuming the forward interest rate will be realized.

The value of a forward rate agreement (FRA) where a fixed rate, R_k, will be received on a principal (L) between times $T_1$and T_2 ,if R_F is the forward rate for the period and R_2 is the spot/zero rate for maturity T_2 is given by:

$$\boldsymbol{V_{FRA,fixed\ rate\ received} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}}$$

The value of FRA where a fixed rate R_k, is paid is

$$\boldsymbol{V_{FRA,fixed\ rate\ paid} = L(R_F - R_K)(T_2 - T_1)e^{-R_2 T_2}}$$

BIONIC TURTLE

Derive the value of the cash flows from a forward rate agreement (FRA) (continued)

Example:

A company enters a 36 v 39 FRA to receive 4.0% ("sell FRA") on $100.0 MM principal for a **three-month period**, 3 years forward. In this way, the company will receive the fixed rate (4.0%) and pay LIBOR.

- If LIBOR is 4.5% in 3 years, company ends up paying to the counterparty who is the buyer receiving the payment at a future value of time t = 3.25 years of $125,000.

$$100{,}000{,}000 \times (4\% - 4.5\%) \times 0.25 = -\$125{,}000$$

- Discounting this to the present value at year 3, we get $\frac{-125{,}000}{(1+4.5\%\times 0.25)} = -123{,}609$.

Hull Example 4.3: Forward Rate Agreement (FRA)

FRA notional		$100,000,000	
Fixed rate [3.0, 3.25]		4.00%	

Year	Realized 3 mo LIBOR		
1.00			
2.00			
3.00	4.50%	($123,609.3943)	*Discounted 0.25 years at 0.045*
3.25		($125,000.0000)	*Notional * (0.040 - 0.045) * 0.25*

BIONIC TURTLE

Derive the value of the cash flows from a forward rate agreement (FRA) (continued)

Another example (Hull Ex 4.4, 10th Edition)

Suppose that the forward LIBOR rate for the period between time 1.5 years and time 2 years in the future is 5% (with semiannual compounding) and that some time ago a company entered into an FRA where it will receive 5.8% (with semiannual compounding) and pay LIBOR on a principal of $100 million for the period. The 2-year risk-free rate is 4% (with continuous compounding). The value of the FRA is

$$100{,}000{,}000 \times (0.058 - 0.050) \times 0.5e^{-0.04\times 2} = \$369{,}200$$

FRA notional	**$100,000,000**	2-yr zero (CC)	**4.00%**
Fixed rate [1.5, 2.0]	**5.80%**	2-yr zero (s.a.)	4.04%

Year	Zero (s.a.)	Forward (s.a.)	Future	Present
				$369,246.5386
0.50				
1.00				
1.50	3.72%			
2.00	4.04%	5.00%	$400,000.0000	

BIONIC TURTLE

Derive the value of the cash flows from a forward rate agreement (FRA) (continued)

There are two notations used for FRAs.

Consider an FRA entered into on Jan 1st. The FRA will expire in six months, at which time the firm will pay the 6-month LIBOR interest rate and receive a fixed rate of 5% (the annualized rate, but only six months' interest will be collected).

The FRA is entered on Jan 1st; the FRA expires on June 30th and the net payment is determined at that time based on the LIBOR rate at that time.

- If LIBOR happens to be 5%, there is no net settlement. If the 6-month LIBOR, in six months' time, happens to be 4%, then firm will receive a payment:

Receive-fixed:	(+) 5% x ½ Year × Notional Principal ($)
Pay-floating:	(-) 4% x ½ Year × Notional Principal ($)
	= (+)1% x ½ Year × Notional Principal ($)

- **The first notation method to describe this swap is given by:**

  6 × 12 5%; i.e., [Term to Expire, Months] × [Term to End of Period Covered by FRA] [Fixed Rate]

- **The second notation method to describe this (same) swap:**

  $FRA_{6,12}$ = 5%

BIONIC TURTLE

Calculate the duration, modified duration and dollar duration of a bond.

Modified duration is a measure of price sensitivity: it is the approximate percentage change in bond price given a 1.0% change in the yield. For example, if the (modified) duration is 3.5 years, then the bond's price will increase approximately 3.5% if the yield drops by 1.0%.

$$\boldsymbol{D = -(\Delta B/B)/\Delta y \rightarrow D\Delta y = -(\Delta B/B)}$$

The related duration measure is **Macaulay Duration.** Macaulay duration is the weighted-average maturity of the bond, where the weight is the present value of the cash flow *as a proportion of the bond's price*. For a bond with price B and cash flows c_i at time t_i giving a continuously compounded yield of y, it is calculated as:

$$\boldsymbol{D = \sum_{i=1}^{n} t_i \frac{c_i\, e^{-y t_i}}{B}}$$

BIONIC TURTLE

Calculate the duration, modified duration and dollar duration of a bond (continued)

Calculation of Macaulay Duration

- **Example:** Consider a 3-year 10% coupon bond with a face value of $100.00 and a yield of 12.0% per annum with **continuous compounding**. Coupon payments of $5.00 are made semiannually.

Hull Table 4.6: Calculation of Macaulay Duration

Face value	$100.00
Semi-annual coupon	10.0%
Yield (continuously comp, CC)	12.0%

Semi-Annual Period (T)	d.f.	Cash flow FV	Cash flow PV	Weight (W)	Time * Weight (T*W)
0.5	0.942	$5.00	$4.71	0.050	0.025
1.0	0.887	$5.00	$4.43	0.047	0.047
1.5	0.835	$5.00	$4.18	0.044	0.066
2.0	0.787	$5.00	$3.93	0.042	0.083
2.5	0.741	$5.00	$3.70	0.039	0.098
3.0	0.698	$105.00	$73.26	0.778	2.333
		$130.00	$94.213	1.000	2.6530
			↑ *Price*		↑ *Mac Duration*

BIONIC TURTLE

Calculate the duration, modified duration and dollar duration of a bond (continued)

Macaulay Duration

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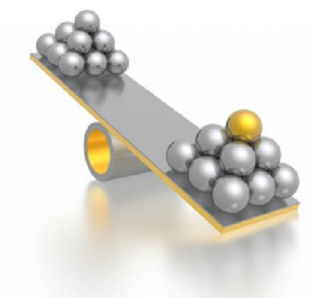


- The present values of the bond's cash flows, using the yield as the discount rate, are shown under PV. For e.g. the present value of the first cash flow is $\$5.00e^{-0.12*0.5} = \$4.71$. This can also be obtained as the cash flow of $5.00 multiplied by the discount factor of 0.942.
- The sum of the PV of all the bond's cash flows returns the bond's price of $94.213.
- The weights are calculated as proportion of the present value of cash flow at each time period divided by the bond price of $94.213. For example, the final cash flow of $105.00 has a present value (PV) of $73.26 = $105.00*EXP(-0.12*3.0). The cash flow, in present value terms, constitutes 0.778 or 77.8% of the bond's price (which is the sum of the present value of all cash flows, by definition). As this cash flow occurs in year 3.0, its contribution to duration is given by 3.0 years * 0.778 = 2.333 years (shown in the final column). The sum of these *weighted-maturities* is the Macaulay duration, which is 2.6530 years.
- These weights are then applied to each time period and summed up to give the bond duration of 2.653 years (the units of both modified and Macaulay duration are years)

Calculate the duration, modified duration and dollar duration of a bond (continued)

Application of duration (Hull Example 4.5)

For the bond in Table 4.6 with a price of $94.213 and duration of 2.653 years, **suppose the yield increases by 10 basis points (0.10%)**. Let's compare the actual bond price due to this yield shock to the estimation given by duration.

Initial bond price, P(0)	$94.21	
Modified duration, D	2.653	If CC, mod = mac
Yield shock, Δy	0.10%	i.e., 10 basis points
Approx ΔP	-$0.2499	
New bond price	$93.96307	
Exact bond price	$93.96343	
Error (%Δ)	-3.79E-06	

- From the definition of duration, we can calculate the change in bond price as

$$\Delta B = -D \times \Delta y \times B = 2.653 \times 0.001 \times 94.21 = -0.2499$$

- Due to the yield shock of 0.10%, the bond price goes down to 93.96307(=94.21-0.2499). This is the *new bond pri*ce as predicted by the duration relationship. The difference between the actual bond price and the one predicted by the duration relationship is negligible as they both are similar up to three decimal places

BIONIC TURTLE

Calculation of Modified Duration

The calculation of Macaulay duration we saw above is based on the assumption that the yield is continuously compounded. Instead, if the yield is compounded with a compounding frequency of m times per year, then the duration relationship is *modified* to:

$$\Delta B = \frac{-D \times \Delta y \times B}{1 + y/m}$$

This change in the duration (which accounts for discrete) compounding of yield is called **Modified duration, D*,** and its relationship to Macaulay duration is given by:

$$D^* = \frac{D}{1 + y/m}$$

such that the duration relationship becomes $\Delta B = -D^* \times \Delta y \times B$

BIONIC TURTLE

Calculate the duration, modified duration and dollar duration of a bond (continued)

Hull Example 4.6

For the same bond in Table 4.6 with a price of 94.213 and a duration of 2.653, if the yield expressed with semiannual compounding is 12.3673%, then the modified duration is calculated as:

$$D^* = \frac{D}{1 + y/m} = \frac{2.653}{1 + 12.3673\%/2} = 2.50$$

So, $\Delta B = -D^* \times \Delta y \times B = -2.50 \times 0.001 \times 94.21 = -0.235$

This means that the new bond price as predicted by the Modified Duration relationship for a change in yield of 0.1% is $93.978 (= 94.213 -0.235).

An exact calculation similar to that in Hull example 4.5 shows that, when the semiannually compounded bond yield increases by 0.10% to 12.4673%, the bond price becomes $93.978. We can see **the modified duration is very accurate for small yield changes.**

Calculate the duration, modified duration and dollar duration of a bond (continued)

Calculation of Dollar Duration

Dollar duration, $D_{\$}$, is the modified duration, D*, multiplied by the bond's current price:

$$\boldsymbol{D_{\$} = D^{*} \times B}$$

Such that the duration relationship becomes $\Delta B = -D_{\$} \times \Delta y$

- **Example:** For the same bond with a price of \$94.213 and a modified duration of 2.50 years, the dollar duration is calculated as:

$$D_{\$} = D^{*} \times \mathrm{B} = 2.50 \times \$94.213 = \$235.80$$

BIONIC TURTLE

Calculate the duration, modified duration and dollar duration of a bond (continued)

Macaulay duration: Weighted average maturity

Modified duration: Sensitivity.

- Modified duration = Mac duration/(1+y/k)

Effective Duration (ED) estimates Modified duration and is the approximate percentage change in bond prices for a percentage change in yield and calculated as:

$$D_e = \frac{Price[if\ yield\ decreases] - Price[if\ yield\ increases]}{2 \times Initial\ Price\ of\ the\ bond\ \times \Delta y}$$

BIONIC TURTLE

Calculate the duration, modified duration and dollar duration of a bond (continued)

	Face value	$100.00		s.a. coupon	10.0%
				s.a. yield	12.0%

Period (T)	d.f.	CF FV	CF PV	Weight (W)	(T*W)
0.5	0.943	$5.00	$4.72	0.050	0.025
1.0	0.890	$5.00	$4.45	0.047	0.047
1.5	0.840	$5.00	$4.20	0.044	0.066
2.0	0.792	$5.00	$3.96	0.042	0.083
2.5	0.747	$5.00	$3.74	0.039	0.098
3.0	0.705	$105.00	$74.02	0.778	2.335
		$130.00	$95.083	1.000	2.6548
					↑

Macaulay (weighted avg maturity), D	2.6548
Modified duration, D(*) = D/(1+y/m) = D/(1+y/2)	2.5046
Effective duration, D(e) = -1/P*[P(+Δy) - P(-Δy)]/(2*Δy)	
Yield shock assumption, Δy	0.50%
P(+Δy)	$93.90
P(-Δy)	$96.28
D(e) = -1/P*[P(+Δy) - P(-Δy)]/(2*Δy)	2.5047

BIONIC TURTLE

Calculate the duration, modified duration and dollar duration of a bond (continued)

Macaulay duration is weighted average maturity of the bond; i.e., the sum of (T*W) in the final column, where (W) is the ratio of the present value of the cash flows to the bond price. The sum of each [maturity * weight] gives the Macaulay duration of 2.6548 years

The **modified duration** is obtained by adjusting the Macaulay duration for semiannual compounding and calculated as:

$$D^* = \frac{D}{1 + y/m} = \frac{2.6548}{1 + 12\%/2} = 2.5046 \text{ years}$$

The **dollar duration** is the modified duration times the bond price and is calculated as:

$$D_\$ = D^* \times \mathrm{B} = 2.5046 \times 95.083 = \$238.145$$

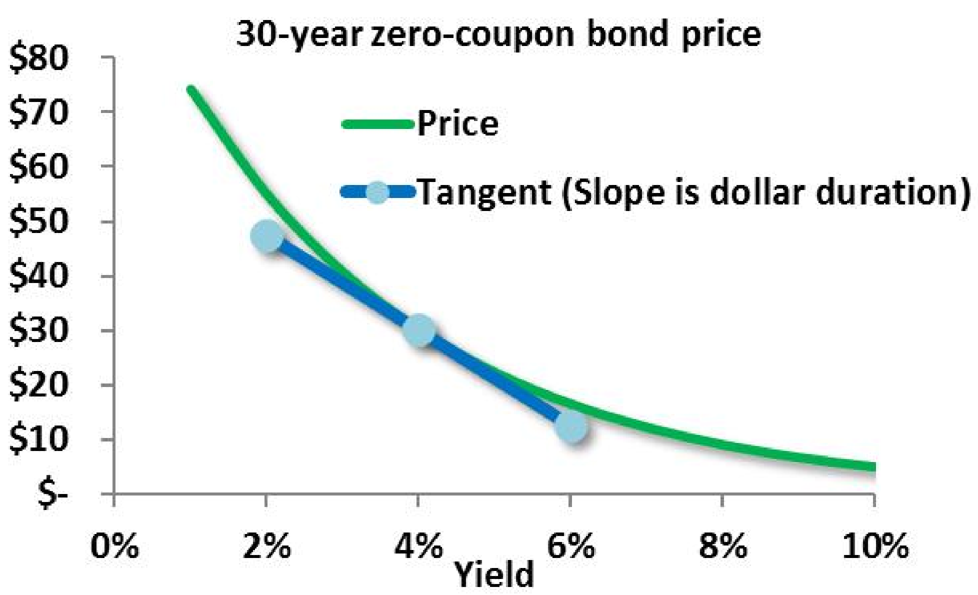
Finally, the **effective duration** is calculated as:

$$\mathrm{D_e} = \frac{\mathrm{Price}_{\Delta y-} - \mathrm{Price}_{\Delta y+}}{2 \times \mathrm{Price}_0 \times \Delta \mathrm{y}} = \frac{96.28 - 93.90}{2 \times 95.083 \times 0.5} = 2.5047 \text{ years}$$

Evaluate the limitations of duration and explain how convexity addresses some of them.

Limitations of duration

By hedging a portfolio to achieve a net duration of zero, exposure is eliminated only with respect to ***small parallel shifts*** in the yield curve. If a portfolio is hedged with respect to duration, it still remains exposed to shifts that are large in magnitude and/or non-parallel.



Evaluate the limitations of duration and explain how convexity addresses some of them (continued)

How convexity can help alleviate the problem

- Convexity, as a function of the second derivative, adjusts for some but not all of the “gap” between duration and the actual price change.
- It does, however, help address the issue of non-parallel shifts and larger changes by more effectively hedging our position against non-linearity.

Note that even with the convexity adjustment, this remains a single-factor model, i.e., the yield to maturity is *the* single factor, with limitations:

- Duration is a first-order linear approximation
- Duration is only accurate for small, parallel shifts in the yield curve
- Convexity adds a term to adjust for the curvature in the price/yield curve
- Convexity is still imprecise
- Both utilize the *Taylor Series approximation*: duration is a function of the first term and convexity is a function of the second term.

Calculate the change in a bond's price given duration, convexity, and a change in interest rates.

Example: Consider a 3-year, 10.0% coupon bond with a face value of $100. It has a **continuously compounded** yield of 12.0% per annum. Now suppose that the yield changes by 10 basis points (0.10%).

Example: Change in a bond's price given duration, convexity, and a change in yield

Face value	$100.00
Semi-annual coupon	10.0%
Yield (continuously comp, CC)	12.0%

Semi-Annual Period (T)	d.f.	Cash flow FV	Cash flow PV	Weight (W)	Time * Weight (T*W)	Time^2 * Weight (T^2*W)
0.5	0.942	$5.00	$4.71	0.050	0.025	0.012
1.0	0.887	$5.00	$4.43	0.047	0.047	0.047
1.5	0.835	$5.00	$4.18	0.044	0.066	0.100
2.0	0.787	$5.00	$3.93	0.042	0.083	0.167
2.5	0.741	$5.00	$3.70	0.039	0.098	0.246
3.0	0.698	$105.00	$73.26	0.778	2.333	6.998
		$130.00	$94.213	1.000	2.6530	7.5700
			↑ *Price*		↑ *Duration*	↑ *Convexity*

Duration plus convexity adjustment		
Initial bond price, P(0)	$94.21	
Modified duration, D	2.653	If CC, mod = mac
Yield shock, Δy	0.10%	i.e., 10 basis points
Approx ΔP	-$0.2496	$= P_0*(-D*\Delta y + 0.5*C*\Delta y^2)$
New bond price	$93.96343	$= P_0 + \Delta P$
Exact bond price	$93.96343	
Error (%Δ)	3.70E-09	approx. zero
Duration-only Error	-3.79E-06	
Ratio of Errors	0.00098	~ eliminates error due to linearity

Calculate the change in a bond's price given duration, convexity, and a change in interest rates (continued)

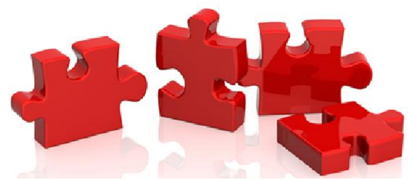
Like we saw earlier in the duration section, from the table, the sum of the PV of all the bond's cash flows give an initial bond price of 94.213 and a bond duration **(Macaulay duration**) of 2.653 years.

- Now we need to convert the Macaulay duration (D) into modified duration (D^*). Note that modified duration can be less than or equal to Macaulay duration, but never greater than! In the special case of continuous compounding, the relationship,

$$D^* = \frac{D}{1+y/m} \text{ reduces to } D^* = D.$$

  Thus, the modified duration will be equal to the Macaulay duration of 2.653 years.

Calculate the change in a bond's price given duration, convexity, and a change in interest rates (continued)



While the duration relationship can be applied for small changes in yields, **convexity** measures the curvature in the yield curve for large yield changes. It is the time squared weighted average of ratio of the present value of the cash flows to the bond price. It is calculated as the sum of T^2 times the weights and found to be 7.57.

- Now, given the yield shock of 10 basis points (0.1%), the change in bond price as given by the duration (D) and convexity (C) relationship is:

$$\Delta B = B\left[-D\Delta y + \frac{1}{2}C(\Delta y)^2\right]$$

$$= 94.21\left[-2.653 * 0.001 + \frac{1}{2} * 7.57 * (0.001)^2\right] = -\$\,0.2496$$

- The change in bond price is added to the initial bond price to obtain the new bond price of $96.96343 (= 94.21 - 0.2496) as predicted by the duration and convexity relationship.

- An exact calculation of the bond, shows that when the bond yield increases by 10 bps to 12.1%, the actual bond price declines to $93.96343 from $94.213.

Compare and contrast the major theories of the term structure of interest rates.

A number of theories have been proposed to explain the shape of the zero (spot rate) curve, including the following three:

1. **Expectations Theory:** This is the simplest (but unrealistic) theory that long-term interest rates should reflect ***only expected future short-term interest rates***. A forward interest rate corresponding to a certain future period is equal to the expected future zero interest rate for that period.

2. **Market Segmentation Theory:** The theory that short, medium & long rates are independent of each other: "*There need be no relationship between short-, medium-, and long-term interest rates. Under the theory, a major investor such as a large pension fund invests in bonds of certain maturity and does not readily switch from one maturity to another. The short-term interest rate is determined by supply and demand in the short-term bond market; medium-term interest rates are determined by supply and demand in the medium-term bond market; and so on.*"

3. **Liquidity Preference**: The Liquidity Preference theory holds that *forward rates are higher than expected future zero rates.* Investors prefer to preserve their liquidity and invest funds for short periods of time. Borrowers prefer to borrow at fixed rates for long periods of time.

BIONIC TURTLE

The End

P1.T3. Financial Markets & Products

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Interest Rates